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ON THE NONSTANDARD REPRESENTATION OF MEASURES

BY

C. WARD HENSON

ABSTRACT. In this paper it is shown that every finitely additive probability measure μ on S which assigns 0 to finite sets can be given a nonstandard representation using the counting measure for some *-finite subset F of $^*S$. Moreover, if μ is countably additive, then F can be chosen so that

$$\int f\,d\mu = \mathrm{st}\left(\frac{1}{\|F\|}\sum_{p\in F} {}^*f(p)\right)$$

for every μ-integrable function f. An application is given of such representations. Also, a simple nonstandard method for constructing invariant measures is presented.

Let S be a set in some set theoretical structure $\mathfrak{M}$ and let $^*S$ be the corresponding set in an enlargement $^*\mathfrak{M}$ of $\mathfrak{M}$. Bernstein and Wattenberg have noted [2] that if F is a *-finite subset of $^*S$, then a finitely additive probability measure μ_F can be defined for all subsets A of S by

$$\mu_F(A) = \mathrm{st}(\|{}^*A \cap F\| / \|F\|). \tag{1}$$

They used this observation as the basis for a nonstandard proof of the theorem, due to Banach [1], which states that Lebesgue measure on $[0, 1]$ can be extended to a totally defined (finitely additive) measure which is invariant under translations (mod 1).

This paper concerns the representation of probability measures as nonstandard counting measures μ_F. Let μ be any finitely additive probability measure which is defined on an algebra $\mathfrak{B}$ of subsets of S and which satisfies $\mu(A) = 0$ for each finite set A in $\mathfrak{B}$. In §1 it is shown that there exists a *-finite subset F of $^*S$ which satisfies $\mu = \mu_F$ on $\mathfrak{B}$. This has the consequence that for any bounded, μ-integrable function f,

$$\int f\,d\mu = \mathrm{st}\left(\frac{1}{\|F\|}\sum_{p\in F} {}^*f(p)\right). \tag{2}$$

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Moreover, if $\mathcal{B}$ is a σ-algebra and μ is countably additive, then F can be chosen so that (2) holds for every μ-integrable function.

Closely related to these results is a nonstandard representation for bounded linear functionals on the space l_∞ of bounded sequences in R, which was given by Robinson [7]. In §2 a straightforward extension of Robinson's result is used to give a nonstandard proof of a convergence result (Theorem 3) for bounded linear functionals on $C(X)$, where X is a compact, Hausdorff space.

Also, in §3 a nonstandard construction of invariant measures is given which yields a particularly simple proof of Banach's extension result for Lebesgue measure.

Preliminaries. The given structure $\mathfrak{M}$ is assumed to have the set R of real numbers as an element (thus also the set N of nonnegative integers). Moreover, the embedding $x \mapsto {}^*x$ of $\mathfrak{M}$ into ${}^*\mathfrak{M}$ is taken to be the identity on R. The standard part of a finite element p of ${}^*R$ is denoted by $\mathrm{st}(p)$. If $p, q \in {}^*R$, then $p =_1 q$ means that $p - q$ is infinitesimal.

For each set S in $\mathfrak{M}$ and each *-finite subset F of ${}^*S$, $\|F\|$ is the "cardinality" of F, in the sense of ${}^*\mathfrak{M}$. That is, if c is the function assigning to each finite subset A of S the cardinality of A, then $\|F\| = {}^*c(F)$. Alternately, $\|F\|$ is the smallest element ω of ${}^*N$ for which there is an internal bijection between F and $\{\omega' \mid \omega' \in {}^*N \text{ and } \omega' < \omega\}$. (For an introduction to the methods of nonstandard analysis see [5], [6] or [8].)

Given a set S, $\mathcal{P}(S)$ is the algebra of all subsets of S. Also, $l_\infty(S)$ is the linear space of all bounded, real valued functions on S, furnished with the sup norm. In this paper μ is a measure on S if it is a nonnegative, finitely additive set function defined on an algebra of subsets of S. If μ is normalized to satisfy $\mu(S) = 1$, then it is a probability measure. The notation $A \triangle B$ will be used for the symmetric difference, $(A \sim B) \cup (B \sim A)$, of two subsets of S.

1. **Nonstandard representations.** Let μ be a probability measure on $\mathcal{P}(S)$ and let ϕ be the linear functional on $l_\infty(S)$ defined by integration with respect to μ. Then ϕ is a positive linear functional of norm 1. Therefore, by the principal result of [7], there exist a *-finite subset F of ${}^*S$ and an internal function λ from F to ${}^*R$ which satisfy

$$\mathrm{st}\left(\sum_{p \in F} |\lambda(p)|\right) = 1$$

and, for each f in $l_\infty(S)$,

$$\phi(f) = \mathrm{st}\left(\sum_{p \in F} \lambda(p)\, {}^*f(p)\right).$$

(Robinson's result [7] only covers the case $S = N$ explicitly, but his argument is easily extended to cover the general case.) Therefore the measure μ has the representation

$$\mu(A) = \mathrm{st}\left(\sum_{p \in {}^*A \cap F} \lambda(p) \right). \tag{3}$$

Theorem 1 below states that, if $\mu(\{s\}) = 0$ for every $s \in S$,[1] then F can be chosen so that μ is represented as in (3), but with every $\lambda(p)$ equal to $1/\|F\|$. That is, $\mu(A) = \mu_F(A)$ for every $A \subset S$.

Theorem 1. *If μ is a probability measure on $\mathcal{P}(S)$ which satisfies $\mu(\{s\}) = 0$ for each $s \in S$, then there is a *-finite set $F \subset {}^*S$ for which $\mu = \mu_F$.*

Proof. Since ${}^*\mathfrak{M}$ is an enlargement of $\mathfrak{M}$, there exists a *-finite subset $\mathfrak{A}$ of ${}^*\mathcal{P}(S)$ which satisfies ${}^*A \in \mathfrak{A}$ for each $A \subset S$. For each internal subset $\mathfrak{F}$ of $\mathfrak{A}$, define

$$E(\mathfrak{F}) = \bigcap \{E \mid E \in \mathfrak{F}\} \cap \bigcap \{{}^*S \sim E \mid E \in \mathfrak{A} \sim \mathfrak{F}\},$$

so that the function taking $\mathfrak{F}$ to $E(\mathfrak{F})$ is internal. Let $\mathfrak{A}' = \{E(\mathfrak{F}) \mid \mathfrak{F}$ is an internal subset of $\mathfrak{A}\}$, so that $\mathfrak{A}'$ is a *-finite set. Moreover, $\mathfrak{A}'$ is a partition of ${}^*S$, and each member of $\mathfrak{A}$ is the union of an internal subset of $\mathfrak{A}'$.

Let $\omega = \|\mathfrak{A}'\|$ and choose $r \in {}^*N$ so that ω^2/r is infinitesimal. For each E in $\mathfrak{A}'$ define $r(E)$ in ${}^*N$ by the inequalities

$$r(E)/r \leq {}^*\mu(E) < (r(E) + 1)/r. \tag{4}$$

Then the function $E \mapsto r(E)$ on $\mathfrak{A}'$ is internal. Moreover, if E is a *-finite element of $\mathfrak{A}'$, then ${}^*\mu(E) = 0$, from which it follows that $r(E) = 0$. Therefore there exists an internal function f which is defined on $\mathfrak{A}'$ and which satisfies: For each E in $\mathfrak{A}'$, $f(E)$ is a *-finite subset of E and $\|f(E)\| = r(E)$.

It will be shown that the set F defined by

$$F = \bigcup \{f(E) \mid E \in \mathfrak{A}'\}$$

satisfies the condition $\mu = \mu_F$. Since the elements of $\mathfrak{A}'$ are pairwise disjoint, the elements of $\{f(E) \mid E \in \mathfrak{A}'\}$ have the same property, and therefore,

$$\|F\| = \sum_{E \in \mathfrak{A}'} r(E).$$

Moreover, since the function ${}^*\mu$ is *-finitely additive,

[1] The added condition on μ is only slightly more restrictive than necessary. Indeed, if F is infinite and $s \in S$, then $\mu_F(\{s\}) \leq \mathrm{st}(1/\|F\|) = 0$. If F is finite, say with k elements, then μ_F is of the form $\mu = k^{-1}(\mu_1 + \cdots + \mu_k)$, where each of the measures μ_j takes on as values only 0 and 1.

$$1 = {}^*\mu({}^*S) = \sum_{E \in \mathfrak{A}'} {}^*\mu(E).$$

Therefore, from the inequalities (4) follows

$$\|F\|/\tau \leq 1 < \|F\|/\tau + \omega/\tau,$$

by summing over E. That is, by the choice of τ, $\omega(\|F\|/\tau - 1)$ is infinitesimal.

Now let A be any element of $\mathfrak{A}$ and let $\mathfrak{F}$ be the collection of E in $\mathfrak{A}'$ which are subsets of A. Therefore A is the union of $\mathfrak{F}$, by the construction of $\mathfrak{A}'$. It follows that

$$\|A \cap F\| = \sum_{E \in \mathfrak{F}} \tau(E), \quad \text{and} \quad {}^*\mu(A) = \sum_{E \in \mathfrak{F}} {}^*\mu(E).$$

Therefore

$$(5) \qquad {}^*\mu(A) - \frac{\|A \cap F\|}{\|F\|} = \sum_{E \in \mathfrak{F}} \left({}^*\mu(E) - \frac{\tau(E)}{\|F\|} \right).$$

But for each E in $\mathfrak{A}'$,

$$\begin{aligned} |{}^*\mu(E) - \tau(E)/\|F\|\,| &\leq |{}^*\mu(E) - \tau(E)/\tau| + |\tau(E)/\tau - \tau(E)/\|F\|\,| \\ &\leq 1/\tau + (\tau(E)/\|F\|)\,|\,\|F\|/\tau - 1\,| \leq 1/\tau + |\,\|F\|/\tau - 1\,|. \end{aligned}$$

Thus (5) implies

$$|{}^*\mu(A) - \|A \cap F\|/\|F\|\,| \leq \omega/\tau + \omega|\,\|F\|/\tau - 1\,|$$

which is infinitesimal. In particular, for each $A \subset S$,

$$\mu(A) = {}^*\mu({}^*A) = \mathrm{st}\,(\|{}^*A \cap F\|/\|F\|) = \mu_F(A).$$

This completes the proof.

While Theorem 1, as stated, applies only to totally defined measures, it is valid for any probability measure μ which is defined on an algebra of subsets of S and which assigns measure 0 to any finite set in its domain. This is because any such measure can be extended to a measure which satisfies the conditions of Theorem 1.

A different nonstandard representation for measures, based on partitions of ${}^*S$ rather than *finite subsets, has been developed and applied by Peter Loeb [5], [6].

Lemma 1. *Let E be any *-finite subset of ${}^*S$ and let F be an internal subset of E which satisfies $\|F\|/\|E\| =_1 1$. Then $\mu_F = \mu_E$ on $\mathcal{P}(S)$ and*

$$\int f\, d\mu_E = \mathrm{st}\left(\frac{1}{\|F\|} \sum_{p \in F} {}^*f(p) \right)$$

for each f in $l_\infty(S)$.

Proof. Let A be any subset of S. Then

$$\big|\, \|{}^*A \cap E\|/\|E\| - \|{}^*A \cap F\|/\|E\| \,\big| \le \|E \sim F\|/\|E\| =_1 0.$$

Therefore

$$\mu_E(A) = \mathrm{st}\,(\|F\|/\|E\| \cdot \|{}^*A \cap F\|/\|F\|) = \mu_F(A).$$

Now let f be any element of $l_\infty(S)$, and define

$$\psi(f) = \mathrm{st}\left(\frac{1}{\|F\|} \sum_{p \in F} {}^*f(p)\right).$$

Then ψ is a bounded linear functional on $l_\infty(S)$. Also, if V is the subspace of $l_\infty(S)$ generated by the characteristic functions, then ψ agrees with the μ_E-integral on V. The fact that V is norm-dense in $l_\infty(S)$ implies that ψ and the μ_E-integral are equal on all of $l_\infty(S)$.

Now let $\mathfrak{B}$ be a σ-algebra of subsets of S and let μ be a countably additive probability measure on $\mathfrak{B}$ which satisfies $\mu(A) = 0$ for each finite set A in $\mathfrak{B}$. There exists an extension $\tilde{\mu}$ of μ to $\mathcal{P}(S)$ which satisfies $\tilde{\mu}(\{s\}) = 0$ for $s \in S$. By Theorem 1, there exists a *-finite subset F of $^*S$ which satisfies $\tilde{\mu} = \mu_F$, and thus $\mu(A) = \mu_F(A)$ for every A in $\mathfrak{B}$.

For any bounded, μ-integrable function f, $\int f\, d\mu = \int f\, d\tilde{\mu}$. Therefore, by Lemma 1,

$$\int f\, d\mu = \mathrm{st}\left(\frac{1}{\|F\|} \sum_{p \in F} {}^*f(p)\right). \tag{6}$$

However, for unbounded, μ-integrable functions (6) may not be true. (Indeed, if f is any unbounded function on S, then F may be chosen satisfying $\mu = \mu_F$ on $\mathfrak{B}$, but such that the sum $\|F\|^{-1} \Sigma_{p \in F}\, {}^*f(p)$ is infinite.) It is possible, nonetheless, to choose F in such a way that (6) is true for every μ-integrable function.

It is convenient to assume that $^*\mathfrak{M}$ is κ-saturated (in the sense of [7]), where κ is any cardinal number greater than the number of functions from S to R. The remainder of this section is devoted to showing that, under this assumption, it is possible to represent μ on $\mathfrak{B}$ in such a way that (6) holds for every μ-integrable function.

Given $n \in N$ and a function f from S to R, define f_n on S by

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Each f_n is a bounded function, and it is measurable whenever f is. Also, if $\omega \in {}^*N$ and $p \in {}^*S$, then

$$ {}^*f_\omega(p) = \begin{cases} {}^*f(p) & \text{if } |{}^*f(p)| \leq \omega, \\ 0 & \text{otherwise.} \end{cases} $$

Lemma 2. *Let E be any *-finite subset of ${}^*S$ which satisfies $\mu = \mu_E$ on $\mathfrak{B}$ and let f be a nonnegative, μ-integrable function. There exists an internal subset F_f of E which satisfies $\|F_f\|/\|E\| =_1 1$ and, for any internal subset F of F_f,*

$$ \frac{\|F\|}{\|E\|} =_1 1 \longrightarrow \int f\,d\mu = \mathrm{st}\left(\frac{1}{\|F\|}\sum_{p\in F} {}^*f(p)\right). $$

Proof. For each $n \in N$, let $A_n = \{x \mid f(x) > n\}$. Then $\{A_n \mid n \in N\}$ is a decreasing chain of sets in $\mathfrak{B}$ and $\bigcap\{A_n \mid n \in N\} = \emptyset$. Thus the sequence $\{\mu(A_n)\}$ decreases monotonically to 0. Since $\mu = \mu_E$ on $\mathfrak{B}$, it follows that for each $\delta > 0$ in R, there exists $n_0 \in N$ which satisfies

$$ n \geq n_0 \longrightarrow \|{}^*A_n \cap E\|/\|E\| < \delta. $$

If ω is an infinite member of ${}^*N$, then ${}^*A_\omega \subset {}^*A_n$, so $\|{}^*A_\omega \cap E\|/\|E\| < \delta$. This shows that for every such ω,

$$ \|{}^*A_\omega \cap E\|/\|E\| =_1 0. \tag{7} $$

Since f is nonnegative, the sequence of integrals $\int f_n\,d\mu$ is increasing. By the monotone convergence theorem, the supremum of this sequence is $\int f\,d\mu$. If $\int f\,d\mu = \int f_n\,d\mu$ for some $n \in N$, then $\mu(A_n) = 0$ and hence

$$ \|E \sim {}^*A_n\|/\|E\| =_1 1. $$

In this case let $F_f = E \sim {}^*A_n$. If $F \subset F_f$ and $\|F\|/\|E\| =_1 1$, then

$$ \int f\,d\mu = \int f_n\,d\mu = \mathrm{st}\left(\frac{1}{\|F\|}\sum_{p\in F} {}^*f(p)\right) $$

since ${}^*f = {}^*f_n$ on F and $\mu_F = \mu_E$.

Therefore it may be assumed that $\int f_n\,d\mu < \int f\,d\mu$ for all $n \in N$. Thus

$$ \frac{1}{\|E\|}\sum_{p\in E} {}^*f_n(p) < \int f\,d\mu $$

for all $n \in N$. It follows that there is an infinite ω in ${}^*N$ which satisfies

$$ \frac{1}{\|E\|}\sum_{p\in E} {}^*f_\omega(p) < \int f\,d\mu. $$

In this case let $F_f = E \sim {}^*A_\omega$, so that $\|F_f\|/\|E\| =_1 1$ by (7). Suppose F is any internal subset of F_f which satisfies $\|F\|/\|E\| =_1 1$. Then, for each $n \in N$,

$$\int f_n \, d\mu \leq \operatorname{st}\left(\frac{1}{\|F\|} \sum_{p \in F} {}^*f(p)\right)$$
$$\leq \operatorname{st}\left(\frac{1}{\|E\|} \sum_{p \in E} {}^*f_\omega(p)\right) = \int f \, d\mu,$$

using Lemma 1 and the fact that ${}^*f = {}^*f_\omega$ on F. By the monotone convergence theorem

$$\operatorname{st}\left(\frac{1}{\|F\|} \sum_{p \in F} {}^*f(p)\right) = \int f \, d\mu,$$

completing the proof.

Theorem 2. *Let $\mathfrak{B}$ be an σ-algebra of subsets of S and let μ be a countably additive probability measure on $\mathfrak{B}$ which satisfies $\mu(A) = 0$ for each finite set A in $\mathfrak{B}$. There exists a *-finite subset F of ${}^*S$ which satisfies $\mu = \mu_F$ on $\mathfrak{B}$ and*

$$\int f \, d\mu = \operatorname{st}\left(\frac{1}{\|F\|} \sum_{p \in F} {}^*f(p)\right)$$

for every μ-integrable function f.

Proof. Let I be the set of nonnegative, μ-integrable functions. Since each μ-integrable function is the difference of two elements of I, it suffices to find an F which satisfies the conditions of the theorem for every f in I. By Theorem 1 (and the remarks following) there exists a *-finite subset E of ${}^*S$ which satisfies $\mu = \mu_E$ on $\mathfrak{B}$. For each $f \in I$, let F_f be a subset of E which satisfies the conditions of Lemma 2. Given $n \in N$ and $f \in I$, define

$$A(n, f) = \{F \mid F \text{ is an internal subset of } F_f \text{ and } \|F\|/\|E\| > n/(n+1)\}.$$

This family of internal sets has cardinality $\operatorname{card}(N \times I)$, which is less than κ. Moreover, the family has the finite intersection property. ($F_{f_1} \cap \cdots \cap F_{f_n}$ is an element of $A(m_1, f_1) \cap \cdots \cap A(m_n, f_n)$ whenever $m_1, \cdots, m_n \in N$ and $f_1, \cdots, f_n \in I$.) Since ${}^*\mathfrak{M}$ is κ-saturated, there exists a *-finite set F which satisfies $F \in A(n, f)$ for every $n \in N$ and $f \in I$ (Theorem 2.7.12 of [5]). That is, $F \subset F_f$ for every $f \in I$, and $\|F\|/\|E\| =_1 1$. It follows by Lemma 2 that F satisfies the conditions of the theorem.

Remark. Theorem 2 is true even if ${}^*\mathfrak{M}$ is not κ-saturated, but the proof of that fact is somewhat more complicated. The proof given here proves the stronger result that F can be chosen as a subset of any given set E which satisfies $\mu = \mu_E$ on $\mathfrak{B}$.

2. **An application.** The following standard result can be proved easily using the Riesz Representation Theorem. The nonstandard proof given here uses the extension to $l_\infty(S)$ of Robinson's representation result [9] instead.

Theorem 3. *Let X be a compact, Hausdorff space, $\{f_n\}$ a sequence in $C(X)$ and ϕ a bounded linear functional on $C(X)$. If $\{f_n\}$ is uniformly bounded on X and converges to 0 pointwise, then $\phi(f_n) \to 0$.*

Proof. Let ϕ be any bounded linear functional on $C(X)$. By the Hahn-Banach theorem, ϕ may be extended to a bounded linear functional $\tilde{\phi}$ on $l_\infty(X)$. By the extension to $l_\infty(X)$ of the principal result of [9], there exist a *-finite subset of $^*X$ and an internal function λ from F into $^*R$ which satisfy

$$\tilde{\phi}(f) = \mathrm{st}\left(\sum_{p\in F} \lambda(p)\,{}^*f(p)\right)$$

for every f in $l_\infty(X)$, and $\Sigma_{p\in F} |\lambda(p)|$ is finite.

Let $\{f_n\}$ be a sequence in $C(X)$ which is uniformly bounded on X by 1, and which converges to 0, pointwise. If $\phi(f_n)$ does not converge to 0, then it may be assumed (by taking a subsequence) that for some $\delta > 0$ in R, $|\phi(f_n)| > \delta$ for every $n \in N$. Let $M = \mathrm{st}\,(\Sigma_{p\in F} |\lambda(p)|) + 1$. For $n \in N$, define

$$A_n = \{x \mid x \in X \text{ and } |f_n(x)| \geq \delta/2M\}.$$

Therefore,

$$\begin{aligned}\delta &< \left|\sum_{p\in F} \lambda(p)\,{}^*f_n(p)\right| \\ &\leq \sum_{p\in {}^*A_n\cap F} |\lambda(p)\,{}^*f_n(p)| + \sum_{p\in F\sim {}^*A_n} |\lambda(p)\,{}^*f_n(p)| \\ &\leq \sum_{p\in {}^*A_n\cap F} |\lambda(p)| + \frac{\delta}{2}.\end{aligned}$$

Thus, for each $n \in N$, $\Sigma_{p\in {}^*A_n\cap F} |\lambda(p)| > \delta/2$.

Now define μ' on $\mathcal{P}(X)$ by

$$\mu'(A) = \mathrm{st}\left(\sum_{p\in {}^*A\cap F} |\lambda(p)|\right)$$

for each $A \subset X$. Then μ' is a measure on $\mathcal{P}(X)$, and $\mu'(A_n) > \delta/2$ for every $n \in N$. It follows that there is an infinite subset K of N such that $\{A_n \mid n \in K\}$ has the finite intersection property (see Lemma 17.9 of [4]). Since $^*\mathfrak{M}$ is an enlargement, there is an element p of $^*X$ which satisfies $|{}^*f_n(p)| \geq \delta/2M$ for all $n \in K$. X is compact, so p is near-standard to some $x \in X$. In particular, ${}^*f_n(p) =_1 f_n(x)$ for every $n \in N$. This implies $|f_n(x)| \geq \delta/2M$ for every $n \in K$, which contradicts the assumption that $f_n(x)$ converges to 0. Therefore $\phi(f_n)$ must converge to 0.

3. **Constructing invariant measures.** Let G be a group of permutations on S, and assume that G satisfies *Følner's condition*:

For each $a_1, \cdots, a_n \in G$ and $k \in N$, there exists a finite set $A \subset G$ which satisfies $\|A \triangle Aa_j\|/\|A\| < 1/(k+1)$ for each $j = 1, \cdots, n$.

To apply the corresponding statement in ${}^*\mathfrak{M}$, let E be a *-finite subset of ${}^*G$ which contains $\{{}^*g \mid g \in G\}$ and let ω be an infinite member of ${}^*N$. Then there is a *-finite set $F \subset {}^*G$ which satisfies $\|F \triangle Fp\|/\|F\| < 1/\omega$ for every $p \in E$. In particular,

$$g \in G \longrightarrow \|F \triangle F{}^*g\|/\|F\| =_1 0. \tag{8}$$

If F satisfies (8), then μ_F is a probability measure on $\mathcal{P}(G)$ and μ_F is invariant under the action of G on itself by right multiplication. The principal result of [3] is, essentially, that the converse holds: If there is such a measure on $\mathcal{P}(G)$, then G satisfies Følner's condition.

Theorem 4. *Let G be a group of permutations of S and let F be a *-finite subset of ${}^*G$ which satisfies* (8). *Let μ be any measure on $\mathcal{P}(S)$ and define $\tilde{\mu}$ by*

$$\tilde{\mu}(A) = \mathrm{st}\left(\frac{1}{\|F\|} \sum_{p \in F} {}^*\mu(p{}^*A)\right)$$

for $A \subset S$. Then $\tilde{\mu}$ is a G-invariant measure on $\mathcal{P}(S)$. Moreover, if $A \subset S$ satisfies $\mu(gA) = \mu(A)$ for every $g \in G$, then $\tilde{\mu}(A) = \mu(A)$.

Proof. Each element of ${}^*G$ is a permutation of ${}^*S$. Thus if A, B are disjoint subsets of S, then $p{}^*A$, $p{}^*B$ are disjoint subsets of ${}^*S$ for each $p \in {}^*G$. Thus ${}^*\mu(p({}^*A \cup {}^*B)) = {}^*\mu(p{}^*A) + {}^*\mu(p{}^*B)$. From this the finite additivity of $\tilde{\mu}$ is immediate.

Given A in $\mathcal{P}(S)$ and g in G,

$$\begin{aligned}
|\tilde{\mu}(gA) - \tilde{\mu}(A)| &=_1 \left|\frac{1}{\|F\|} \sum_{p \in F} ({}^*\mu(p{}^*g{}^*A) - {}^*\mu(p{}^*A))\right| \\
&\le \frac{1}{\|F\|} \sum_{p \in F \triangle F{}^*g} {}^*\mu(p{}^*A) \\
&\le \mu(S) \cdot \|F \triangle F{}^*g\|/\|F\| =_1 0.
\end{aligned}$$

Therefore $\tilde{\mu}(gA) = \tilde{\mu}(A)$, so that $\tilde{\mu}$ is G-invariant.

Finally, suppose A is a subset of S which satisfies $\mu(gA) = \mu(A)$ for every $g \in G$. Then ${}^*\mu(p{}^*A) = {}^*\mu({}^*A)$ for every $p \in {}^*G$. Therefore

$$\tilde{\mu}(A) = \mathrm{st}\left(\frac{1}{\|F\|} \sum_{p \in F} {}^*\mu({}^*A)\right) = \mu(A).$$

To prove Banach's extension result, let G be the group of all translations (mod 1) of $[0, 1]$, and let μ be any extension of Lebesgue measure to $\mathcal{P}([0, 1])$. It is well known, and easy to prove using the decomposition theorem for finitely generated abelian groups, that every abelian group satisfies Følner's condition. Since G is abelian, Theorem 4 can be applied to obtain a G-invariant measure $\tilde{\mu}$ on $\mathcal{P}([0, 1])$. If A is a Lebesgue measurable subset of $[0, 1]$, then $\mu(gA) = \mu(A)$ for every $g \in G$. Theorem 4 thus asserts that $\tilde{\mu}(A) = \mu(A)$; that is, $\tilde{\mu}$ is an extension of Lebesgue measure.

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DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM, NORTH CAROLINA 27706